

# EXISTENCE OF A MEROMORPHIC EXTENSION OF SPECTRAL ZETA FUNCTIONS ON FRACTALS

BENJAMIN A. STEINHURST AND ALEXANDER TEPLYAEV

**ABSTRACT.** We investigate the existence of the meromorphic extension of the spectral zeta function of the Laplacian on self-similar fractals using the classical results of Kigami and Lapidus (based on the renewal theory) and new results of Hambly and Kajino based on the heat kernel estimates and other probabilistic techniques. We also formulate conjectures which hold true in the examples that have been analyzed in the existing literature.

There has been many works in mathematical physics, analysis, and probability on fractals studying the spectral and heat kernel asymptotics of various Laplacians on fractal sets, see [1, 4, 14, 18, 22, 29, 37, and references therein], and it is possible that fractal spaces may provide useful models for the study of quantum gravity [3]. In particular, on many fractals the short time asymptotics of the partition function are not given by a power function alone but in many cases the power function is corrected by a multiplicatively periodic function. This behavior has been observed in [27, 17, 16, 2] for finitely ramified fractals, and [1, 18, 22] extend the class of fractals for which one can expect the log-periodic oscillations in the short time heat kernel asymptotics.

In this paper we investigate a related question of existence of a meromorphic continuation of the spectral zeta function, which has found many profitable uses in physics [15, 28] (e.g. Casimir effect [9, 11]). If the Weyl ratio for the eigenvalue counting function is a multiplicatively periodic function, up to a smaller order term (as proved by Kigami and Lapidus in [27]), then the spectral zeta function can be expected to be meromorphic in some region to the left of  $d_S$  where  $d_S$  is the spectral dimension of the underlying Laplacian. We discuss how new results by Hambly and Kajino [18, 22, 23] can be applied to obtain a meromorphic continuation of the spectral zeta function of the Laplacian on certain fractals, such as finitely ramified symmetric fractals and the Sierpinski carpets. Furthermore, if the partition function is decomposed into a sum of power functions times a multiplicatively periodic terms, and an exponentially decreasing term (with no other terms), then the spectral

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zeta function is meromorphic over the whole complex plane (this is done, in particular, in relation to the recent papers [1] concerning the physical implications the existence of meromorphic continuations).

If the Laplace operator  $L$  has a discrete spectrum with eigenvalues  $\lambda_l$ , repeated according to their multiplicities, then the spectral zeta function of  $L$  is given by

$$(1) \quad \zeta(s, \gamma) = \sum_{l=1}^{\infty} (\lambda_l + \gamma)^{-s/2}$$

whenever the series converges absolutely. The use of  $s/2$  instead of  $s$  is not essential, but is preceded in the cited literature and only changes the results by a scaling factor of 2. Recall that the partition function of a non-negative self-adjoint operator  $L$  is  $Z_L(t) = \text{Tr}(e^{-tL})$ , which decays exponentially for large  $t$  in the case of a discrete spectrum with no or excluded zero eigenvalue. By applying the inverse Mellin transform ([12, 13]) to  $Z_L(t)$ , we have

$$(2) \quad \zeta(2s, \gamma) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} Z_L(t) e^{-\gamma t} dt.$$

We consider Laplacians on self-similar compact sets  $F$ , which are connected metric space with injective contraction maps  $\{\psi_j\}_{j=1}^N$  such that  $\psi_j : F \rightarrow F$  and  $F = \bigcup_{j=1}^N \psi_j(F)$ . For the sake of simplicity, we only consider the unique probability self-similar measure  $\mu$  on  $F$  with equal weights, that is  $\mu(\psi_j(F)) = 1/N$ . On a self-similar set, in addition to a self-similar metric and measure, one could ask what it means for a Dirichlet form to be self-similar. We assume existence of the following decomposition of a local regular Dirichlet form  $\mathcal{E}$  on  $F$

$$(3) \quad \mathcal{E}(f, g) = \rho_F \sum_{i=1}^N \mathcal{E}(f \circ \psi_i, g \circ \psi_i),$$

and note that the effect of applying  $\mathcal{E}$  to  $f \circ \psi_i$  is to make  $\mathcal{E}$  to acting only on  $\psi_i(F)$ . With appropriate boundary conditions and domain, the Laplacian  $\Delta$  is defined in a weak sense by  $\mathcal{E}(f, g) = \int_F f \Delta g d\mu$ .

The constant  $\rho_F$  is called the energy rescaling factor or conductance scaling factor. Its reciprocal  $r = \frac{1}{\rho_F}$  is the resistance scaling factor. In a number of papers the value of  $\rho_F$  is explicitly calculated for various finitely ramified fractals (see [4, 8, 20, 25, 36, 38] and references therein), however for infinitely ramified fractals the only example where the exact energy scaling is obtained is [7, 33, 34] (information on eigenvalues and eigenfunctions can be found in [24, 32]). For generalized Sierpinski carpets,  $\rho_F$  can be estimated and the uniqueness can

be proved (see [5, 19, 6] and references therein). Note that the spectral dimension is given by

$$d_S = \frac{2 \log(N)}{\log(\tau)} = \frac{2 \log(N)}{\log(\rho_F N)} = 2 \frac{d_f}{d_w} = 2 \frac{d_0}{d_w}$$

where  $d_f = d_0$  is the Hausdorff dimension and  $d_w$  is the so-called walk dimension (see Lemma 2 and the related work [35] of Strichartz on the spectral dimension). The Laplacian scaling factor  $\tau = \rho_F N$  is also known as the time scaling factor.

We say that a self-similar set  $K$  is finitely ramified and fully symmetric if the following three conditions hold:

- (1) there exists a finite subset  $V_0$  of  $K$  such that  $\psi_j(K) \cap \psi_k(K) = \psi_j(V_0) \cap \psi_k(V_0)$  for  $j \neq k$  (this intersection may be empty);
- (2) if  $v_0 \in V_0 \cap \psi_j(K)$  then  $v_0$  is the fixed point of  $\psi_j$ ;
- (3) there is a group  $\mathfrak{G}$  of isometries of  $K$  that has a doubly transitive action on  $V_0$  and is compatible with the self-similar structure  $\{\psi_j\}_{j=1}^N$ , which means ([31, Proposition 4.9] and also [4, 21, 30]) that for any  $j$  and any  $g \in \mathfrak{G}$  there exists  $k$  such that  $g \circ \psi_j = \psi_k$ .

Moreover, a fully symmetric finitely ramified self-similar set  $K$  is a p.c.f. self-similar set if and only if for any  $v_0 \in V_0$  there is a unique  $j$  such that  $v_0 \in \psi_j(K)$  [21, 31].

Our first result is the following theorem, which improves the main result in [37] (this result is related to the gaps in the spectrum, see [21]).

**Theorem 1.** *On any fully symmetric p.c.f. fractal, as defined above, the spectral zeta function with  $\gamma = 0$  has a meromorphic continuation to  $\operatorname{Re}(s) > -\epsilon$  for some positive  $\epsilon$  with at most two sequences of poles, also called spectral dimensions, at  $\operatorname{Re}(s) = d_S$  and  $\operatorname{Re}(s) = 0$ .*

*Proof.* It is easy to see that the spectral zeta function is analytic for  $\operatorname{Re}(s) > d_S$  and there is a simple pole at  $s = d_S$ . From the results in [4, 37] we obtain that there exists a meromorphic continuation to the half-plane  $\operatorname{Re}(s) > -\epsilon$  with finitely many sequences of poles in  $0 \leq \operatorname{Re}(s) \leq d_S$ . In addition, [22, Theorem 7.7 and Corollary 7.8] and Lemma 1 applied with  $\gamma = 0$  imply that there are no poles in  $0 < \operatorname{Re}(s) < d_S$ . Note that according to [22, Definition 2.10 and Definition 6.8], a fully symmetric p.c.f. fractal has zero-dimensional rational boundary and so there are heat kernel estimates (see [26]). Thus only possible sequences of poles are at  $\operatorname{Re}(s) = d_S$  and  $\operatorname{Re}(s) = 0$ , which completes the proof.  $\square$

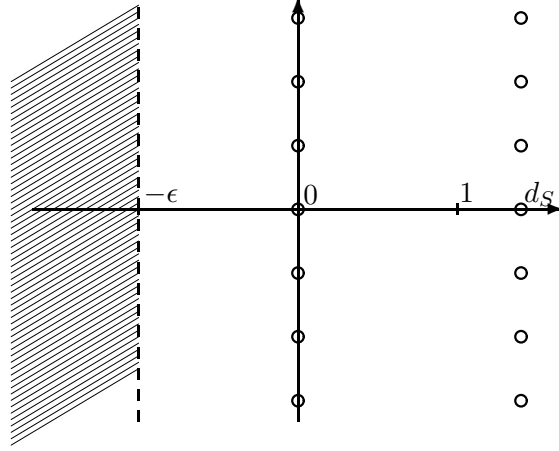


FIGURE 1. Complex spectral dimensions of the Laplacian on a fully symmetric finitely ramified fractal. Unpublished results of Kajino imply that in fact there is a meromorphic continuation for all  $s \in \mathbb{C}$ .

**Theorem 2.** *For any intersection type finite self-similar structures (see [22, Theorem 7.7 and Corollary 7.8]), including fully symmetric p.c.f. fractals, nested fractals and generalized Sierpinski carpets, the spectral zeta function associated to the self-similar Laplacian has a meromorphic extension to beyond the spectral dimension, at least to the half-plane  $\text{Re}(s) > 2\frac{d_\partial}{d_w}$ . Moreover, the spectral zeta function satisfies the following functional equation for  $\gamma > -c_4$  and  $\text{Re}(s) > 2\frac{d_\partial}{d_w}$*

$$(4) \quad \frac{d}{d\gamma} \zeta(s, \gamma) = -\gamma \zeta(s + 2, \gamma).$$

*The poles of  $\zeta(s, 0)$  are located, in the region  $\text{Re}(s) > 2\frac{d_\partial}{d_w}$ , at  $ds + \frac{4\pi in}{\log(\tau)}$ . When  $\gamma \neq 0$  they are located at  $ds - 2m + \frac{4\pi in}{\log(\tau)}$  for  $m \geq 0$ .*

*Proof.* Similarly to the previous result, this is implied by Lemma 1 and [18] and [22, Theorem 7.7 and Corollary 7.8]. The same argument as in the proof of Lemma 1 for differentiating under the integral for  $I_1(s, \gamma)$  applies also to  $I_2(s, \gamma)$  and  $I_3(s, \gamma)$  with the same functional equation. The location of the poles when  $\gamma = 0$  is observed from directly summing the series in (13) when  $\gamma = 0$ . When  $\gamma \neq 0$  the poles when  $m = 0$  are obtained from the same estimate, and the translations of poles by  $2m$  is forced by the functional equation.  $\square$

**Remark 1.** *Note that in the case of the standard Sierpinski carpet  $2\frac{d_f}{d_w} - 2$  will be less than  $2\frac{d_\partial}{d_w}$  so that there are no extra poles in the*

right half-plane. In fact, the spectral zeta function associated to the self-similar Laplacian on the Sierpinski carpet has a meromorphic extension to the whole complex plane because, by the work [23, Theorem 4, in preparation] of Kajino, the conditions of Theorem 2 are satisfied. Moreover, the same is true for a large classes of fractals, such as nested fractals and generalized Sierpinski carpets where the values of  $d_k$  have a geometric meaning. For example  $d_0 = d_f$  and  $d_k$  is the Minkowski dimension of the co-dimension  $k$  faces of the carpet and  $d_d = 0$  i.e. is the Minkowski dimension of the single point that is a co-dimension  $d$  face of the carpet.

**Conjecture 1.** *We conjecture that for fully symmetric finitely ramified fractals, even without heat kernel estimates, the spectral zeta function with  $\gamma = 0$  has a meromorphic continuation to  $\mathbb{C}$  with at most two sequences of poles, also called spectral dimensions, at  $\text{Re}(s) = d_S$  and  $\text{Re}(s) = 0$ . This applies for the usual Dirichlet Laplacian, and for the Neumann Laplacian if the zero eigenvalue is excluded.*

**Conjecture 2.** *We conjecture that if for generalized Sierpinski carpets the possible poles of the spectral zeta function with real part  $2\frac{d_k}{d_w}$ , with  $k = 1, \dots, d-1$ , are actually removable singularities because there are different self-similar (graph-directed) structures that yield the same Laplacian operator. This applies for the usual Neumann Laplacian if the zero eigenvalue is excluded. For the Dirichlet Laplacian the dimension of the boundary will play a role in the spectral asymptotics.*

The two dimensional standard Sierpinski carpet can be realized by two such structures and in this case it is conjectured that there are only two sequences of poles one at  $\text{Re}(s) = 2\frac{d_f}{d_w}$  and the other at  $\text{Re}(s) = 0$ . This has been observed in the case of some Laakso spaces in [7].

**Lemma 1.** *Suppose that  $d_\partial < d_f$  and for  $t < 1$*

$$(5) \quad c_1 t^{-d_\partial/d_w} \leq t^{-d_f/d_w} G\left(\log \frac{1}{t}\right) - Z_L(t) \leq c_2 t^{-d_\partial/d_w}$$

*where  $G$  is a periodic function bounded above and away from zero with period  $\log(\tau)$ , while for  $t \geq 1$  there exist  $c_3, c_4 \geq 0$  such that*

$$(6) \quad |Z_L(t)| \leq c_3 e^{-c_4 t}.$$

*Then, for any  $\gamma > -c_4$ ,  $\zeta(s, \gamma)$  has a meromorphic continuation for  $\text{Re}(s) > 2\frac{d_\partial}{d_w}$ .*

*Proof.* Note that inverse Mellin transformations are linear so that we may transform each of the asymptotics separately. By assumption here

exist bounded measurable functions  $B(t)$  and  $C(t)$  such that for  $t < 1$

$$(7) \quad Z_L(t) = t^{-d_f/d_w} G\left(\log \frac{1}{t}\right) + B(t)t^{-d_\partial/d_w}$$

and for  $t \geq 1$

$$(8) \quad Z_L(t) = C(t)e^{-c_4 t}.$$

Then the Mellin transform of  $Z_L(t)e^{-\gamma t}$  is

$$(9)$$

$$\zeta(2s, \gamma) = \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} t^{-d_f/d_w} G\left(\log \frac{1}{t}\right) e^{-\gamma t} B(t) dt$$

$$(10)$$

$$+ \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} B(t) t^{-d_\partial/d_w} e^{-\gamma t} dt$$

$$(11)$$

$$+ \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} C(t) e^{-c_4 t} e^{-\gamma t} dt = I_1(s, \gamma) + I_2(s, \gamma) + I_3(s, \gamma).$$

Since  $B(t)$  and  $C(t)$  are bounded functions, they do not contribute to the divergence or convergence of these integrals and may be ignored without loss of generality. Note that for all  $\gamma \in \mathbb{R}$ ,  $I_1(s, \gamma)$  converges if  $\operatorname{Re}(s) > \frac{d_f}{d_w}$  and  $I_2(s, \gamma)$  converges if  $\operatorname{Re}(s) > \frac{d_\partial}{d_w}$ , while  $I_3(s, \gamma)$  converges for all  $s \in \mathbb{C}$  and  $\gamma > -c_4$ . It suffices to show that  $I_1(s, \gamma)$  can be meromorphically extended to  $\operatorname{Re}(s) > \frac{d_\partial}{d_w}$ .

Let  $\log(\tau)$  be the period of  $G(T)$  so that  $G(\log(\tau)T)$  has period 1 in the variable  $T$ . Using the change of variables  $t \mapsto \tau^T$  then

$$(12) \quad I_1(s, \gamma) = \frac{\log(\tau)}{\Gamma(s)} \int_{-\infty}^0 (\tau^T)^{s-d_f/d_w} e^{-\gamma \tau^T} G(\log(\tau)T) dT$$

$$(13) \quad = \frac{\log(\tau)}{\Gamma(s)} \sum_{p=-\infty}^{-1} \int_p^{p+1} \tau^{T(s-d_f/d_w)} e^{-\gamma \tau^T} G(\log(\tau)T) dT.$$

The issue of convergence is only at  $T = -\infty$  and thus the integral  $I_1(s, \gamma)$  will converge if the summation converges absolutely. This can be established by using the Taylor series in  $\gamma$ . Moreover if the integral  $I_1(s, \gamma)$  converges for a specific pair  $(s, \gamma)$  it will be analytic in  $s$  in some small neighborhood of  $s$  for that value of  $\gamma$ . Note that if  $s = x + iy$  with  $x > \frac{d_f}{d_w}$ , then

$$|I_1(s, \gamma)| \leq \frac{\log(\tau)}{|\Gamma(s)|} G^* \max\{1, e^{-\gamma}\} \sum_{p=-\infty}^{-1} \tau^{(p+1)(x-d_f/d_w)}$$

where  $G^* = \int_p^{p+1} G(\log(\tau)T) dt$  which is independent of  $p$  by the periodicity of  $G(\log(\tau)T)$ . This last sum is geometric in  $p$  so if  $x$  is replaced by  $s = x + iy$  this bound on  $|I_1(s, \gamma)|$  has a meromorphic extension to the complex plane with poles at  $s = \frac{d_f}{d_w} + \frac{2i\pi n}{\log(\tau)}$  for all fixed  $\gamma \in \mathbb{R}$ .

The integrand in (12) is for  $\gamma > -c_4$  smooth in  $\gamma$  and bounded by a  $T$ -integrable function independent of  $\gamma$  in the region  $Re(s) > \frac{d_f}{d_w}$ . It is then possible to pass a derivative with respect to  $\gamma$  inside of the integral and obtain (4). Repeating this argument it is possible to find  $\frac{d^l}{d\gamma^l} I_1(s, \gamma)$  iteratively for  $\gamma > -c_4$  and  $Re(s) > \frac{d_f}{d_w} - l$ . Since  $I_1(s, \gamma)$  is analytic in a right half-plane this implies that it varies smoothly in  $\gamma \in (-c_4, \infty)$  and  $I_1(s, \gamma)$  can be recovered by integrating  $\frac{d^l}{d\gamma^l} I_1(s, \gamma)$  over  $\gamma$ . This not only gives a meromorphic extension of  $I_1(s, \gamma)$  to  $Re(s) > \frac{d_f}{d_w}$  but also to the whole complex plane (see next lemma for the use of this fact). Notice that if  $Re(s) > \frac{d_f}{d_w} - l$  this definition is consistent with the definition of  $I_1(s, \gamma)$  for  $Re(s) > \frac{d_f}{d_w}$ .  $\square$

**Lemma 2.** *Suppose that for  $t < 1$*

$$(14) \quad Z_L(t) = \sum_{k=0}^d t^{-\frac{d_k}{d_w}} G_k \left( \log \frac{1}{t} \right) + O \left( \exp \left( -ct^{-\frac{1}{d_w-1}} \right) \right)$$

where the  $G_k$  are periodic functions bounded above, and for  $t \geq 1$  there exist  $c_5, c_6 \geq 0$  such that

$$(15) \quad -c_5 e^{-c_6 t} \leq Z_L(t) \leq c_3 e^{-c_6 t}.$$

Then  $\zeta(s, \gamma)$  has a meromorphic continuation to the complex plane for  $\gamma > -c_6$ .

*Proof.* The technique for handling the  $I_1$  term in Lemma 1 is repeated for each of the  $t^{-\frac{d_k}{d_w}} G_k \left( \log \frac{1}{t} \right)$  terms with their respective periods, giving their Mellin transforms and meromorphic continuations. Each of these meromorphic functions have poles at  $s = \frac{d_k}{d_w} - 2m + \frac{2i\pi n}{\log(\tau)}$  for  $n \in \mathbb{Z}$  and  $m \geq 0$ . There is no analogue of the  $I_2(s, \gamma)$  term. The  $I_3(s, \gamma)$  term of Lemma 1 is now replaced with a term of the form

$$(16) \quad \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} e^{(-ct^{-\frac{1}{d_w-1}})} e^{-\gamma t} dt,$$

which converges if the same integral from 0 to  $\infty$  converges. It is known that the inverse Mellin transform of such an exponential term is the product of a complex exponential with base  $c$  and a shifted Gamma function is meromorphically extendable to the whole plane with well known poles in the left half-plane. The existence of the

meromorphic extension of the integral of  $t^{s-1}e^{-c_6t-\gamma t}$  is standard and also is precisely the argument of Lemma 1 concerning the  $I_3$  term. The sum of these meromorphic function has discrete poles in a finite number of towers that do not accumulate thus  $\zeta(s, \gamma)$  is meromorphic with complex dimensions whose real parts are given by  $\frac{d_k}{d_w}$ , where  $d_0 = d_f$  and  $G_0$  is not identically zero. Thus a meromorphic extension of  $\zeta(2s, \gamma)$  can be given for the complex plane provided that  $\gamma > -c_6$ .  $\square$

**Corollary 1.** *Under the assumptions of Lemma 2, the functional equation of Theorem 2 holds in the whole complex plane.*

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DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NY 14853-4201 [HTTP://WWW.MATH.CORNELL.EDU/~STEINHURST/](http://www.math.cornell.edu/~steinhurst/)  
*E-mail address:* [steinhurst@math.cornell.edu](mailto:steinhurst@math.cornell.edu)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONNECTICUT, STORRS, CT, 06269 [HTTP://WWW.MATH.UCONN.EDU/~TEPLYAEV/](http://www.math.uconn.edu/~teplyaev/)  
*E-mail address:* [teplyaev@uconn.edu](mailto:teplyaev@uconn.edu)